A NOTE ON REGULAR SEQUENCES AND THE KOSZUL COMPLEX

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Abstract. In the present paper, we shall prove several basic theorems for regular sequences and the Koszul complex, in a more general setting.

INTRODUCTION

Let $A$ be a Noetherian ring, $I$ be an ideal of $A$ and $M$ be a finite $A$-module. First of all, let us recall several basic facts of $M$-regular sequences and $I$-depth of $M$.

We say that $a_1, \ldots , a_r \in I$ is an $M$-regular sequence (or simply $M$-sequence) in $I$ of length $r$, if the following conditions are satisfied:

(1) for each $1 \leq i \leq r$, $a_i$ is not a zero-divisor on $M/(a_1, \ldots , a_{i-1})M$;
(2) $M \neq (a_1, \ldots , a_r)M$.

If, moreover, there exists no $b \in I$ such that $a_1, \ldots , a_r, b$ is $M$-regular, then $a_1, \ldots , a_r$ is said to be a maximal $M$-regular sequence in $I$.

Theorem A (cf. Theorem 2.1). Let $A$ be a Noetherian ring, $M$ be a finite $A$-module and $I$ be an ideal of $A$ such that $IM \neq M$. Then all maximal $M$-sequences in $I$ have the same length $n$, furthermore $n$ given by

$$n = \inf \{ i \mid \text{Ext}^i_A(A/I, M) \neq 0 \}.$$}

The integer $n$ above is called $I$-depth of $M$ and denoted by $\text{depth}(I, M)$ (if $IM = M$, we set $\text{depth}(I, M) = \infty$).

We shall prove that Theorem A holds in a more general setting pointed out by Prof. Matsumura in [2]. In detail the hypothesis that $M$ is a finite $A$-module can be weakened to the statement that $M$ is a finite $B$-module for a homomorphism $A \rightarrow B$ of Noetherian rings.

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1. PRELIMINARIES

Let us begin with the following lemma. It is seem to be simple but the technical core of this paper.

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Lemma 1.1. Let $M$ be a finite $B$-module for a homomorphism $\phi: A \to B$ of Noetherian rings. Let $I$ be an ideal of $A$ consisting entirely of zero-divisors of $M$. If we set $\text{Ass}_B(M) = \{P_1, \ldots, P_r\}$ and $p_i = P_i \cap A$, then $p_i \in \text{Ass}_A(M)$ for all $i$ and there exists $p_j \in \text{Ass}_A(M)$ such that $I \subseteq p_j$.

Proof. It is easy to see that $P_i = \text{ann}_B(m)$ for some $m \in M$ implies $p_i = \text{ann}_A(m)$ by the definition of module structures. Hence $p_i \in \text{Ass}_A(M)$ for all $i$.

For any $x \in I$, there exists $m \in M$ such that $xm = 0$, $m \neq 0$. Since $xm = \phi(x)m = 0$, $\phi(x)$ is a zero-divisor of $M$ as a $B$-module. Thus $\phi(x)$ is contained in $\bigcup_{i=1}^r P_i$ and $\phi(x) \in P_j$ for some $1 \leq j \leq r$. So $x \in p_j$. Therefore $I \subseteq p_j$. \hfill $\square$

Remark 1.1. Note that according to [1] (9.A), we have, more precisely

$$\text{Ass}_A(M) = \{p_1, \ldots, p_r\}.$$  

By virtue of Lemma 1.1, we have the following generalized theorem:

Theorem 1.2. Let $M$ be a finite $B$-module for a homomorphism $A \to B$ of Noetherian rings and $I$ an ideal of $A$ with $IM \neq M$. Let $n > 0$ be an integer. Then the following are equivalent:

1. $\text{Ext}_A^i(N, M) = 0$ for all $i < n$ and for any finite $A$-module $N$ with $\text{Supp}(N) \subseteq V(I)$;

2. $\text{Ext}_A^i(A/I, M) = 0$ for all $i < n$;

3. There exists a finite $A$-module $N$ with $\text{Supp}(N) = V(I)$ such that $\text{Ext}_A^i(N, M) = 0$ for all $i < n$;

4. There exists an $M$-sequence $a_1, \ldots, a_n$ of length $n$ in $I$.

Proof. (1) $\implies$ (2): Since $\text{Supp}_A(A/I) = V(\text{ann}_A(A/I)) = V(I)$ and $A/I$ is a finite $A$-module, $\text{Ext}_A^i(A/I, M) = 0$ for all $i < n$.

(2) $\implies$ (3) is obvious.

(3) $\implies$ (4): If $I$ consists only of zero-divisors of $M$, then by using Lemma 1.1, there exists an associated prime $p \supseteq I$ of $M$. Hence there exists the exact sequence

$$0 \longrightarrow A/p \longrightarrow M.$$  

Localising at $p$, we get the injective $A_p$-homomorphism $\phi: k \longrightarrow M_p$ where $k$ is the residue field of $A_p$. Now $p$ is contained in $V(I) = \text{Supp}_A(N)$, so that $N_p \neq 0$, and hence by Nakayama’s lemma, $N \otimes_A k = N_p/pN_p \neq 0$. Thus $N \otimes_A k$ is non zero vector space over $k$ and hence its dual space $\text{Hom}_k(N \otimes_A k, k)$ is not zero. We take $0 \neq \psi \in \text{Hom}_k(N \otimes_A k, k)$ and consider a sequence of $A_p$-homomorphisms

$$N_p \longrightarrow N \otimes_A k \longrightarrow k \overset{\psi}{\longrightarrow} k \overset{\phi}{\longrightarrow} M_p.$$  

Here the first arrow is a surjective homomorphism because it is induced by a canonical map $A_p \longrightarrow k$. Hence, $\text{Hom}_{A_p}(N_p, M_p) \neq 0$. The left-hand side is equal to $(\text{Hom}_A(N, M))_p$ because $N$ is a finite $A$-module, so that $\text{Ext}_A^0(N, M) = \text{Hom}_A(N, M) \neq 0$. But this contradicts (3). Hence $I$ contains an $M$-regular element $f$. $M/fM \neq 0$ because $M/I M \neq 0$. If $n = 1$ then we are done. Assume $n > 1$, we set $M_1 = M/fM$, then the exact sequence

$$0 \longrightarrow M \overset{f}{\longrightarrow} M \longrightarrow M_1 \longrightarrow 0$$

is a counterexample. Therefore $n > 1$.
leads to the long exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ext}_A^0(N, M) & \longrightarrow & \text{Ext}_A^0(N, M) & \longrightarrow & \text{Ext}_A^0(N, M_1) \\
& \longrightarrow & \text{Ext}_A^1(N, M) & \longrightarrow & \text{Ext}_A^1(N, M) & \longrightarrow & \text{Ext}_A^1(N, M_1) \\
& & \longrightarrow & \text{Ext}_A^2(N, M) & \longrightarrow & \cdots.
\end{array}
\]

By the assumption, we have

\[
\text{Ext}_A^0(N, M_1) = \cdots = \text{Ext}_A^{n-2}(N, M_1) = 0.
\]

So \(\text{Ext}_A^i(N, M_1) = 0\) for all \(i < n - 1\). Since \(M_1/I^1 M = M/IM \neq 0\), by inductive hypothesis, there exists an \(M_1\)-sequence \(f_2, \ldots, f_n\) of length \(n - 1\) in \(I\). Hence \(f_1, f_2, \ldots, f_n\) is an \(M\)-sequence of length \(n\) in \(I\). Thus we get the assertion.

(4) \(\Rightarrow\) (1): We show this by induction on \(n\). For \(n = 1\), there exists an \(M\)-regular element \(f_1 \in I\). We set \(M_1 = M/f_1 M\), then the short exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & M & \xrightarrow{f_1} & M & \longrightarrow & M_1 & \longrightarrow & 0
\end{array}
\]

leads the exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}_A(N, M) & \xrightarrow{f_1} & \text{Hom}_A(N, M).
\end{array}
\]

Since \(N\) is a finite \(A\)-module, \(\text{Supp}_A(N) = V(\text{ann}_A(N))\) so that we have \(\sqrt{\text{ann}_A(N)} \supseteq \sqrt{I} \supseteq I\). Hence \(f_1^i \text{Hom}_A(N, M) = 0\) for a sufficiently large \(l > 0\). Therefore we have \(\text{Ext}_A^0(N, M) = \text{Hom}_A(N, M) = 0\), because \(f_1\) is injective. For \(n > 1\), then \(f_2, \ldots, f_n\) is an \(M_1\)-sequence of length \(n - 1\). By inductive hypothesis, we have \(\text{Ext}_A^i(N, M_1) = 0\) for all \(i < n - 1\). Hence the short exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & M & \xrightarrow{f_1} & M & \longrightarrow & M_1 & \longrightarrow & 0
\end{array}
\]

leads the long exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ext}_A^0(N, M) & \xrightarrow{f_1} & \text{Ext}_A^0(N, M) & \longrightarrow & \text{Ext}_A^0(N, M_1) = 0 \\
& \longrightarrow & \text{Ext}_A^1(N, M) & \xrightarrow{f_1} & \text{Ext}_A^1(N, M) & \longrightarrow & \text{Ext}_A^1(N, M_1) = 0 \\
& & \longrightarrow & \text{Ext}_A^2(N, M) & \xrightarrow{f_1} & \cdots.
\end{array}
\]

Thus we have the exact sequence

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ext}_A^i(N, M) & \xrightarrow{f_1} & \text{Ext}_A^i(N, M) & \longrightarrow & \text{Ext}_A^i(N, M_1) = 0
\end{array}
\]

for all \(i < n\). Since \(I \subseteq \sqrt{\text{ann}_A(N)}\) and \(\text{Ext}_A^i(N, M)\) is annihilated by elements of \(\text{ann}_A(N)\), we have \(f_1^i \text{Ext}_A^i(N, M) = 0\) for a sufficiently large \(l > 0\). Therefore \(\text{Ext}_A^i(N, M) = 0\) for all \(i < n\).

\[
\square
\]

2. MAIN RESULTS

Now we come to the main result of the paper.

**Theorem 2.1.** Let \(M\) be a finite \(B\)-module for a homomorphism \(A \rightarrow B\) of Noetherian rings and \(I\) an ideal of \(A\) with \(|IM| \neq 0\). Then, the length of a maximal \(M\)-sequence in \(I\) is the same length \(n\), furthermore \(n\) is determined by

\[
\text{Ext}_A^i(A/I, M) = 0 \text{ for all } i < n \text{ and } \text{Ext}_A^0(A/I, M) \neq 0.
\]

**Proof.** Let us begin with the following claim:
Claim 2.2. Let $a_1, \ldots, a_n \in I$ be an $M$-sequence. Then there exists a sequence of injective homomorphisms

$$
0 \longrightarrow \text{Hom}_A(A/I, M_n) \longrightarrow \text{Ext}_A^1(A/I, M_{n-1}) \longrightarrow \cdots
$$

$$
\cdots \longrightarrow \text{Ext}_A^{n-1}(A/I, M_1) \longrightarrow \text{Ext}_A^n(A/I, M),
$$

where $M_i = M/(a_1, \ldots, a_i)M$.

Proof. By $a_n$ is $M_{n-1}$-regular, we have the short exact sequence

$$
0 \longrightarrow M_{n-1} \xrightarrow{a_n} M_{n-1} \longrightarrow M_n \longrightarrow 0.
$$

This leads the long exact sequence

$$
\cdots \longrightarrow \text{Hom}_A(A/I, M_{n-1}) \longrightarrow \text{Hom}_A(A/I, M_n) \longrightarrow \text{Ext}_A^1(A/I, M_{n-1}) \longrightarrow \cdots.
$$

Since $a_n$ is an $M_{n-1}$-sequence of length 1, by Theorem 1.2, $\text{Hom}_A(A/I, M_{n-1}) = 0$. Thus

$$
\text{Hom}_A(A/I, M_n) \longrightarrow \text{Ext}_A^1(A/I, M_{n-1}).
$$

Next $a_{n-1}$ is $M_{n-2}$-regular, the short exact sequence

$$
0 \longrightarrow M_{n-2} \xrightarrow{a_{n-1}} M_{n-2} \longrightarrow M_{n-1} \longrightarrow 0.
$$

leads the long exact sequence

$$
\cdots \longrightarrow \text{Ext}_A^1(A/I, M_{n-2}) \longrightarrow \text{Ext}_A^1(A/I, M_{n-1}) \longrightarrow \text{Ext}_A^2(A/I, M_{n-2}) \longrightarrow \cdots.
$$

Since $a_n, a_{n-1}$ is an $M_{n-2}$-sequence of length 2, by Theorem 1.2, $\text{Ext}_A^1(A/I, M_{n-2}) = 0$, so that

$$
\text{Ext}_A^1(A/I, M_{n-1}) \longrightarrow \text{Ext}_A^2(A/I, M_{n-2}).
$$

Similarly proceeding this way, we get our assertion. □

Let $a_1, \ldots, a_n \in I$ be a maximal $M$-sequence. If $\text{Ext}_A^n(A/I, M) = 0$ by the above claim, we have $\text{Hom}_A(A/I, M_n) = 0$. By Theorem 1.2, there exists an $M_n$-regular element $a_{n+1} \in I$ and so we have an $M$-sequence $a_1, \ldots, a_{n+1}$ in $I$, but this contradicts to the maximality of $a_1, \ldots, a_n$. Thus we get the assertion. □

Finally, let us see that a basic result on the Koszul complex can be generalized:

Theorem 2.3. Let $M$ be a finite $B$-module for a homomorphism $A \to B$ of Noetherian rings and $I = (y_1, \ldots, y_n)$ an ideal of $A$ such that $IM \neq M$. If we set

$$
q = \sup \{i \mid H_i(y, M) \neq 0\},
$$

then any maximal $M$-sequence in $I$ has length $n - q$.

Proof. Let $x_1, \ldots, x_s$ be a maximal $M$-sequence in $I$. We prove this by induction on $s$. For $s = 0$, every element of $I$ is a zero-divisor of $M$. By Lemma 1.1, there exists $p \in \text{Ass}_A(M)$ such that $I \subseteq p$. Then, there exists $0 \neq \xi \in M$ such that $p = \text{ann}_A(\xi)$, and hence $I\xi = 0$. Thus $\xi \in H_n(y, M)$ because $H_n(y, M) = \{\xi \in M \mid y_1\xi = \cdots = y_n\xi = 0\}$, so that $q = n$ and the assertion holds in this case.

For $s > 0$, we set $M_1 = M/x_1M$. Then, the short exact sequence

$$
0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M_1 \longrightarrow 0
$$

leads the long exact sequence (see [3, Proposition 1.6.11]),

$$
\cdots \longrightarrow H_{i-1}(y, M) \xrightarrow{x_1} H_i(y, M) \longrightarrow H_i(y, M_1) \longrightarrow H_{i-1}(y, M_1) \longrightarrow \cdots.
$$
By [2, Theorem 16.4], \( \ker (H_i(y, M)) = 0 \), so that
\[
\ker (H_i(y, M)) \to H_i(y, M_1) = \text{im } x_1 = 0,
\]
\[
\text{im } (H_i(y, M_1)) \to H_{i-1}(y, M)) = \ker x_1 = H_{i-1}(y, M).
\]
Thus we have the short exact sequence
\[
0 \to H_i(y, M) \to H_i(y, M_1) \to H_{i-1}(y, M) \to 0
\]
for all \( i \). If \( H_{q+1}(y, M_1) = 0 \) then \( H_q(y, M) = 0 \) from the above exact sequence. But this is a contradiction, hence \( H_{q+1}(y, M_1) \neq 0 \). Since \( H_{i-1}(y, M) = H_i(y, M) = 0 \) for all \( i > q+1 \), so that \( H_i(y, M_1) = 0 \). This means that \( q+1 = \text{sup} \{ i \mid H_i(y, M_1) \neq 0 \} \). Since \( M_1 \) is a finite \( B \)-module and \( M_1 \neq IM_1 \), by the inductive hypothesis, \( s - 1 = n - (q + 1) \). Therefore \( s = n - q \). □

References


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