A NOTE ON REGULAR SEQUENCES AND THE KOSZUL COMPLEX

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Abstract. In the present paper, we shall prove several basic theorems for regular sequences and the Koszul complex, in a more general setting.

Introduction

Let A be a Noetherian ring, I be an ideal of A and M be a finite A-module. First of all, let us recall several basic facts of M-regular sequences and I-depth of M.

We say that $a_1, \dots, a_r \in I$ is an M-regular sequence(or simply M-sequence) in I of length r, if the following conditions are satisfied:

(1) for each $1 \leqslant i \leqslant r$, a_i is not a zero-divisor on $M/(a_1, \dots, a_{i-1})M$;

(2) $M \neq (a_1, \dots, a_r)M$.

If, moreover, there exists no $b \in I$ such that $a_1, \dots, a_r, b$ is M-regular, then $a_1, \dots, a_r$ is said to be a maximal M-regular sequence in I.

Theorem A (cf. Theorem 2.1). *Let A be a Noetherian ring, M be a finite A-module and I be an ideal of A such that $IM \neq M$. Then all maximal M-sequences in I have the same length n, furthermore n given by*

$$n = \inf\{\, i \mid \mathrm{Ext}^i_A(A/I, M) \neq 0 \,\}.$$

The integer n above is called I-depth of M and denoted by $\mathrm{depth}(I, M)$ *(if $IM = M$, we set* $\mathrm{depth}(I, M) = \infty$*).*

We shall prove that Theorem A holds in a more general setting pointed out by Prof. Matsumura in [2]. In detail the hypothesis that M is a finite A-module can be weakened to the statement that M is a finite B-module for a homomorphism $A \longrightarrow B$ of Noetherian rings.

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1. preliminaries

Let us begin with the following lemma. It is seem to be simple but the technical core of this paper.

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Lemma 1.1. *Let M be a finite B-module for a homomorphism $\phi : A \to B$ of Noetherian rings. Let I be an ideal of A consisting entirely of zero-divisors of M. If we set $\mathrm{Ass}_B(M) = \{P_1, \ldots, P_r\}$ and $\mathfrak{p}_i = P_i \cap A$, then $\mathfrak{p}_i \in \mathrm{Ass}_A(M)$ for all i and there exists $\mathfrak{p}_j \in \mathrm{Ass}_A(M)$ such that $I \subseteq \mathfrak{p}_j$.*

Proof. It is easy to see that $P_i = \mathrm{ann}_B(m)$ for some $m \in M$ implies $\mathfrak{p}_i = \mathrm{ann}_A(m)$ by the definition of module structures. Hence $\mathfrak{p}_i \in \mathrm{Ass}_A(M)$ for all i.

For any $x \in I$, there exists $m \in M$ such that $xm = 0$, $m \neq 0$. Since $xm = \phi(x)m = 0$, $\phi(x)$ is a zero-divisor of M as a B-module. Thus $\phi(x)$ is contained in $\bigcup_{i=1}^r P_i$ and $\phi(x) \in P_j$ for some $1 \leqslant j \leqslant r$. So $x \in \mathfrak{p}_j$. Therefore $I \subseteq \mathfrak{p}_j$. $\square$

Remark 1.1. Note that according to [1] (9.A), we have, more precisely

$$\mathrm{Ass}_A(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}.$$

By virtue of Lemma 1.1, we have the following generalized theorem:

Theorem 1.2. *Let M be a finite B-module for a homomorphism $A \to B$ of Noetherian rings and I an ideal of A with $IM \neq M$. Let $n > 0$ be an integer. Then the following are equivalent:*

(1) $\mathrm{Ext}_A^i(N, M) = 0$ *for all $i < n$ and for any finite A-module N with* $\mathrm{Supp}(N) \subseteq V(I)$*;*

(2) $\mathrm{Ext}_A^i(A/I, M) = 0$ *for all $i < n$;*

(3) *There exists a finite A-module N with* $\mathrm{Supp}(N) = V(I)$ *such that* $\mathrm{Ext}_A^i(N, M) = 0$ *for all $i < n$;*

(4) *There exists an M-sequence $a_1, \ldots, a_n$ of length n in I.*

Proof. (1) $\Longrightarrow$ (2): Since $\mathrm{Supp}_A(A/I) = V(\mathrm{ann}_A(A/I)) = V(I)$ and A/I is a finite A-module, $\mathrm{Ext}_A^i(A/I, M) = 0$ for all $i < n$.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (4): If I consists only of zero-divisors of M, then by using Lemma 1.1, there exists an associated prime $\mathfrak{p} \supseteq I$ of M. Hence there exists the exact sequence

$$0 \longrightarrow A/\mathfrak{p} \longrightarrow M.$$

Localising at $\mathfrak{p}$, we get the injective $A_\mathfrak{p}$-homomorphism $\phi : k \longrightarrow M_\mathfrak{p}$ where k is the residue field of $A_\mathfrak{p}$. Now $\mathfrak{p}$ is contained in $V(I) = \mathrm{Supp}_A(N)$, so that $N_\mathfrak{p} \neq 0$, and hence by Nakayama's lemma, $N \otimes_A k = N_\mathfrak{p}/\mathfrak{p}N_\mathfrak{p} \neq 0$. Thus $N \otimes_A k$ is non zero vector space over k and hence its dual space $\mathrm{Hom}_k(N \otimes_A k, k)$ is not zero. We take $0 \neq \psi \in \mathrm{Hom}_k(N \otimes_A k, k)$ and consider a sequence of $A_\mathfrak{p}$-homomorphisms

$$N_\mathfrak{p} \longrightarrow N \otimes_A k \xrightarrow{\ \psi\ } k \xrightarrow{\ \phi\ } M_\mathfrak{p}.$$

Here the first arrow is a surjective homomorphism because it is induced by a canonical map $A_\mathfrak{p} \longrightarrow k$. Hence, $\mathrm{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) \neq 0$. The left-hand side is equal to $(\mathrm{Hom}_A(N, M))_\mathfrak{p}$ because N is a finite A-module, so that $\mathrm{Ext}_A^0(N, M) = \mathrm{Hom}_A(N, M) \neq 0$. But this contradicts (3). Hence I contains an M-regular element f. $M/fM \neq 0$ because $M/IM \neq 0$. If $n = 1$ then we are done. Assume $n > 1$, we set $M_1 = M/fM$, then the exact sequence

$$0 \longrightarrow M \xrightarrow{\ f\ } M \longrightarrow M_1 \longrightarrow 0$$

leads to the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \operatorname{Ext}^0_A(N,M) \longrightarrow \operatorname{Ext}^0_A(N,M) \longrightarrow \operatorname{Ext}^0_A(N,M_1) \\ &\longrightarrow \operatorname{Ext}^1_A(N,M) \longrightarrow \operatorname{Ext}^1_A(N,M) \longrightarrow \operatorname{Ext}^1_A(N,M_1) \\ &\longrightarrow \operatorname{Ext}^2_A(N,M) \longrightarrow \cdots . \end{aligned}$$

By the assumption, we have

$$\operatorname{Ext}^0_A(N,M_1) = \cdots = \operatorname{Ext}^{n-2}_A(N,M_1) = 0.$$

So $\operatorname{Ext}^i_A(N,M_1) = 0$ for all $i < n-1$. Since $M_1/IM_1 = M/IM \neq 0$, by inductive hypothesis, there exists an M_1-sequence $f_2,\dots,f_n$ of length $n-1$ in I. Hence $f_1,f_2,\dots,f_n$ is an M-sequence of the length n in I. Thus we get the assertion.

(4) $\Longrightarrow$ (1): We show this by induction on n. For $n = 1$, there exists an M-regular element $f_1 \in I$. We set $M_1 = M/f_1M$, then the short exact sequence

$$0 \longrightarrow M \xrightarrow{f_1} M \longrightarrow M_1 \longrightarrow 0$$

leads the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(N,M) \xrightarrow{f_1} \operatorname{Hom}_A(N,M).$$

Since N is a finite A-module, $\operatorname{Supp}_A(N) = V(\operatorname{ann}_A(N))$ so that we have $\sqrt{\operatorname{ann}_A(N)} \supseteq \sqrt{I} \supseteq I$. Hence $f_1^l \operatorname{Hom}_A(N,M) = 0$ for a su ciently large $l > 0$. Therefore we have $\operatorname{Ext}^0_A(N,M) = \operatorname{Hom}_A(N,M) = 0$, because f_1 is injective. For $n > 1$, then $f_2,\dots,f_n$ is an M_1-sequence of length $n-1$. By inductive hypothesis, we have $\operatorname{Ext}^i_A(N,M_1) = 0$ for all $i < n-1$. Hence the short exact sequence

$$0 \longrightarrow M \xrightarrow{f_1} M \longrightarrow M_1 \longrightarrow 0$$

leads the long exact sequence

$$\begin{aligned} 0 &\longrightarrow \operatorname{Ext}^0_A(N,M) \xrightarrow{f_1} \operatorname{Ext}^0_A(N,M) \longrightarrow \boxed{\operatorname{Ext}^0_A(N,M_1) = 0} \\ &\longrightarrow \operatorname{Ext}^1_A(N,M) \xrightarrow{f_1} \operatorname{Ext}^1_A(N,M) \longrightarrow \boxed{\operatorname{Ext}^1_A(N,M_1) = 0} \\ &\longrightarrow \operatorname{Ext}^2_A(N,M) \xrightarrow{f_1} \cdots . \end{aligned}$$

Thus we have the exact sequence

$$0 \longrightarrow \operatorname{Ext}^i_A(N,M) \xrightarrow{f_1} \operatorname{Ext}^i_A(N,M)$$

for all $i < n$. Since $I \subseteq \sqrt{\operatorname{ann}_A(N)}$ and $\operatorname{Ext}^i_A(N,M)$ is annihilated by elements of $\operatorname{ann}_A(N)$, we have $f_1^l \operatorname{Ext}^i_A(N,M) = 0$ for a su ciently large $l > 0$. Therefore $\operatorname{Ext}^i_A(N,M) = 0$ for all $i < n$. □

2. MAIN RESULTS

Now we come to the main result of the paper.

Theorem 2.1. *Let M be a finite B-module for a homomorphism $A \to B$ of Noetherian rings and I an ideal of A with $IM \neq M$. Then, the length of a maximal M-sequence in I is the same length n, furthermore n is determined by*

$$\operatorname{Ext}^i_A(A/I,M) = 0 \ \ for\ all\ i < n\ and\ \ \operatorname{Ext}^n_A(A/I,M) \neq 0.$$

Proof. Let us begin with the following claim:

Claim 2.2. *Let $a_1, \dots, a_n \in I$ be an M-sequence. Then there exists a sequence of injective homomorphisms*

$$0 \longrightarrow \operatorname{Hom}_A(A/I, M_n) \longrightarrow \operatorname{Ext}^1_A(A/I, M_{n-1}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \operatorname{Ext}^{n-1}_A(A/I, M_1) \longrightarrow \operatorname{Ext}^n_A(A/I, M),$$

where $M_i = M/(a_1, \dots, a_i)M$.

Proof. By a_n is M_{n-1}-regular, we have the short exact sequence

$$0 \longrightarrow M_{n-1} \xrightarrow{a_n} M_{n-1} \longrightarrow M_n \longrightarrow 0.$$

This leads the long exact sequence

$$\cdots \to \operatorname{Hom}_A(A/I, M_{n-1}) \to \operatorname{Hom}_A(A/I, M_n) \to \operatorname{Ext}^1_A(A/I, M_{n-1}) \to \cdots.$$

Since a_n is an M_{n-1}-sequence of length 1, by Theorem 1.2, $\operatorname{Hom}_A(A/I, M_{n-1}) = 0$. Thus

$$\operatorname{Hom}_A(A/I, M_n) \to \operatorname{Ext}^1_A(A/I, M_{n-1}).$$

Next a_{n-1} is M_{n-2}-regular, the short exact sequence

$$0 \longrightarrow M_{n-2} \xrightarrow{a_{n-1}} M_{n-2} \longrightarrow M_{n-1} \longrightarrow 0.$$

leads the long exact sequence

$$\cdots \to \operatorname{Ext}^1_A(A/I, M_{n-2}) \to \operatorname{Ext}^1_A(A/I, M_{n-1}) \to \operatorname{Ext}^2_A(A/I, M_{n-2}) \to \cdots.$$

Since a_n, a_{n-1} is an M_{n-2}-sequence of length 2, by Theorem 1.2, $\operatorname{Ext}^1_A(A/I, M_{n-2}) = 0$, so that

$$\operatorname{Ext}^1_A(A/I, M_{n-1}) \to \operatorname{Ext}^2_A(A/I, M_{n-2}).$$

Similarly proceeding this way, we get our assertion. □

Let $a_1, \dots, a_n \in I$ be a maximal M-sequence. If $\operatorname{Ext}^n_A(A/I, M) = 0$ by the above claim, we have $\operatorname{Hom}_A(A/I, M_n) = 0$. By Theorem 1.2, there exists an M_n-regular element $a_{n+1} \in I$ and so we have an M-sequence $a_1, \dots, a_{n+1}$ in I, but this contradicts to the maximality of $a_1, \dots, a_n$. Thus we get the assertion. □

Finally, let us see that a basic result on the Koszul complex can be generalized:

Theorem 2.3. *Let M be a finite B-module for a homomorphism $A \to B$ of Noetherian rings and $I = (y_1, \dots, y_n)$ an ideal of A such that $IM \neq M$. If we set*

$$q = \sup\{i \mid H_i(\underline{y}, M) \neq 0\},$$

then any maximal M-sequence in I has length $n - q$.

Proof. Let $x_1, \dots, x_s$ be a maximal M-sequence in I. We prove this by induction on s. For $s = 0$, every element of I is a zero-divisor of M. By Lemma 1.1, there exists $\mathfrak{p} \in \operatorname{Ass}_A(M)$ such that $I \subseteq \mathfrak{p}$. Then, there exists $0 \neq \xi \in M$ such that $\mathfrak{p} = \operatorname{ann}_A(\xi)$, and hence $I\xi = 0$. Thus $\xi \in H_n(\underline{y}, M)$ because $H_n(\underline{y}, M) = \{\xi \in M | y_1\xi = \cdots = y_n\xi = 0\}$, so that $q = n$ and the assertion holds in this case.

For $s > 0$, we set $M_1 = M/x_1M$. Then, the short exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M_1 \longrightarrow 0$$

leads the long exact sequence (see [3, Proposition 1.6.11]),

$$\cdots \longrightarrow H_i(\underline{y}, M) \xrightarrow{x_1} H_i(\underline{y}, M) \longrightarrow H_i(\underline{y}, M_1)$$
$$\longrightarrow H_{i-1}(\underline{y}, M) \xrightarrow{x_1} H_{i-1}(\underline{y}, M) \longrightarrow H_{i-1}(\underline{y}, M_1) \longrightarrow \cdots.$$

By [2, Theorem 16.4], $IH_i(\underline{y}, M) = 0$, so that

$$\operatorname{Ker}\big(H_i(\underline{y}, M) \longrightarrow H_i(\underline{y}, M_1)\big) = \operatorname{Im} x_1 = 0,$$
$$\operatorname{Im}\big(H_i(\underline{y}, M_1) \longrightarrow H_{i-1}(\underline{y}, M)\big) = \operatorname{Ker} x_1 = H_{i-1}(\underline{y}, M).$$

Thus we have the short exact sequence

$$0 \longrightarrow H_i(\underline{y}, M) \longrightarrow H_i(\underline{y}, M_1) \longrightarrow H_{i-1}(\underline{y}, M) \longrightarrow 0$$

for all i. If $H_{q+1}(\underline{y}, M_1) = 0$ then $H_q(\underline{y}, M) = 0$ from the above exact sequence. But this is a contradiction, hence $H_{q+1}(\underline{y}, M_1) \neq 0$. Since $H_{i-1}(\underline{y}, M) = H_i(\underline{y}, M) = 0$ for all $i > q+1$, so that $H_i(\underline{y}, M_1) = 0$. This means that $q+1 = \sup\{ i \mid H_i(\underline{y}, M_1) \neq 0\}$. Since M_1 is a finite B-module and $M_1 \neq IM_1$, by the inductive hypothesis, $s - 1 = n - (q+1)$. Therefore $s = n - q$. □

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