Oscillation theorems for a class of fourth order nonlinear functional differential equations

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Abstract. We consider the oscillatory behavior of fourth order functional differential equations

$$(\text{A}) \qquad (|y''(t)|^{\alpha-1}y'')'' + q(t)|y(t+\sin t)|^{\beta-1}y(t+\sin t) = 0,$$

where α, β are positive constants, $q(t)$ is a positive continuous functions on $[0,\infty)$.

1. Introduction

The aim of this paper is to investigate the oscillatory behaivor of fourth order quasi-linear differential equation with deviating arguments of the form

$$(\text{A}) \qquad (|y''(t)|^{\alpha-1}y''(t))'' + q(t)|y(t+\sin t)|^{\beta-1}y(t+\sin t) = 0, \qquad t \geq 0$$

where α and β are positive constants and $q : [0,\infty) \to (0,\infty)$ is a continuous function.

By a solution of (A) we mean a function $y : [T_y, \infty) \to \mathbb{R}$ which is twice conttinuously differentiable together with $|y''|^{\alpha-1}y''$ and satisfies the equation (A) at all sufficiently large t. These solutions which vanish in a neighborhood of infinity will be excluded from our consideration. A solution is called oscillatory if it has a sequence of zeros clustering at ∞ and nonoscillatory otherwise.

Our origin depends heavily on oscillaiton theory of fourth order quasilinear ordinary differential equations

$$(\text{B}) \qquad (|y''(t)|^{\alpha-1}y''(t))'' + q(t)|y(t)|^{\beta-1}y(t) = 0, \qquad t \geq 0$$

developed by Wu [1], in conjuction with a comparison principle which enables us to deduce oscillation of an equation of the form (A) from that of a similar equation with a different functional argument. Consequently, we can to show the existence of classes of equations of the form (A) for which oscillation criteria can be established.

2. Main results

The objective is to establish criteria for the oscillation of all solutions of the equation (A). We make use of the oscillation results of Wu [1] for the associated ordinary differential equation (B).

Theorem A. *Let $\alpha \geq 1 > \beta$. All solutions of* (B) *are oscillatory if and only if*

$$(2.1) \qquad \int_0^\infty t^{(2+\frac{1}{\alpha})\beta} q(t)dt = \infty.$$

Our idea is to deduce oscillation criteria for (A) from Theorem A by means of the following two lemmas which relate the oscillation of the equation

$$(2.2) \qquad (|u''(t)|^{\alpha-1}u'')'' + F(t, u(h(t))) = 0$$

to that of the equations

$$(2.3) \qquad (|v''(t)|^{\alpha-1}v''(t))'' + G(t, v(k(t))) = 0$$

and

$$(2.4)\qquad (|w''(t)|^{\alpha-1}w''(t))'' + \frac{l'(t)}{h'(h^{-1}(l(t)))}F(h^{-1}(l(t)), w(l(t))) = 0.$$

Here $\alpha > 0$ is a constant and $h,\ k,\ l$ are continuously differential functions on $[0,\infty)$ such that

$$h'(t) > 0,\quad k'(t) > 0,\quad l'(t) > 0,\quad \lim_{t\to\infty} h(t) = \lim_{t\to\infty} k(t) = \lim_{t\to\infty} l(t) = \infty,$$

and that $F,\ G$ are continuous functions on $[0,\infty)\times\mathbb{R}$ such that $uF(t,u) \geq 0,\ uG(t,u) \geq 0$ and $F(t,u),\ G(t,u)$ are nondecreasing in u for any fixed $t \geq 0$. Naturally, h^{-1} denotes the inverse function of h.

Lemma 2.1. *Suppose that* $h(t) \geq k(t)$, $t \geq 0$ *and*

$$(2.5)\qquad F(t,x)\operatorname{sgn} x \geq G(t,x)\operatorname{sgn} x, \qquad (t,x) \in [0,\infty)\times\mathbb{R}.$$

If all the solutions of (2.3) are oscillatory, then so are all the solutions of (2.2).

Lemma 2.2. *Suppose that* $l(t) \geq h(t)$ *for* $t \geq 0$. *If all the solutions of* (2.4) *are oscillatory, then so are all the solutions of* (2.2).

The proof of Lemma 2.1 and 2.2 will be omitted.

We give a sufficient condition for all solutions of (A) to be oscillatory.

Theorem 2.3. *Let* $\alpha \geq 1 > \beta$. *Suppose that there exists a continuously differential function* $h : [0,\infty) \to (0,\infty)$ *such that* $h'(t) > 0$, $\lim_{t\to\infty} h(t) = \infty$, *and* $t \geq h(t)$ *for all large* t. *If*

$$(2.6)\qquad \int_0^\infty (h(t))^{(2+\frac{1}{\alpha})\beta} q(t)dt = \infty,$$

then all solutions of (A) *are oscillatory.*

Proof. Let us consider the equation

$$(2.7)\qquad (|z''(t)|^{\alpha-1}z''(t))'' + q(t)|z(h(t))|^{\beta-1}z(h(t)) = 0,$$

$$(2.8)\qquad (|w''(t)|^{\alpha-1}w''(t))'' + \frac{q(h^{-1}(t))}{h'(h^{-1}(t))}|w(t)|^{\beta-1}w(t) = 0.$$

Since

$$\int_0^\infty t^{(2+\frac{1}{\alpha})\beta}\frac{q(h^{-1}(t))}{h'(h^{-1}(t))}dt = \int_0^\infty (h(\tau))^{(2+\frac{1}{\alpha})\beta} q(\tau)d\tau = \infty$$

by (2.6), Theorem A implies that all solutions of (2.8) are oscillatory. Application of Lemma 2.2 then that all solutions of (2.7) are oscillatory, and the conclusion of the theorem follows from comparison of (A) with (2.8) by means of Lemma 2.1. This completes the proof.

References

[1] Wu Fentao, Nonoscillatory solutions of fourth order quasilinear differential equations, Funkcial. Ekvac. **45** (2002), pp. 71–88.

(2004. 11. 24 受理)