Block idempotents of the crossed Burnside ring of S_5

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1 Crossed Burnside rings

This note is based on [OY 03]. The notation and terminology used in this note are standard. The reader may refer to any books on finite group theory, the representation theory of finite groups and [OY 01]. Let $\mathcal{O}$ be a complete discrete valuation ring with quotient field $\mathbb{K}$ and residue field k of characteristic p. Let P be a p-subgroup of G and e a p-block of $\mathcal{O}C_G(P)$. We define an idempotent

$$[P,e]=\sum_{\theta\in e} e_{P,\theta},$$

where $e_{P,\theta}$'s are primitive idempotents of $\mathbb{K}\otimes X\Omega(G,\mathcal{P},G^c)$ indexed by P. We write $\mathcal{B}(G)$ as the complete set of representatives of conjugacy classes of $\{[P,e]\mid P\in\mathcal{P},\ e\in Bl(C_G(P))\}$. Let $\sim_p$ be the equivalence relation on $\mathcal{B}(G)$ defined by

$$[P,e]\sim_p[Q,f]\Longleftrightarrow \omega_{P,\eta}(x)\equiv\omega_{Q,\theta}(x)\,\forall x\in X\Omega(G,\mathcal{P},G^c),$$

where $\eta\in e$ and $\theta\in f$. We pointed out the following theorems in [OY 03] without proof.

(1.1) Theorem. *Let $Q\leq P\leq G$ and Q, $P\in\mathcal{P}$. Let (P,e) and (Q,f) be Brauer pairs of G. Then $[P,e]\sim_p[Q,f]$ if and only if $(P,e)\supseteq(Q,f)$.*

(1.2) Theorem. *Let b be a block of $\mathcal{O}G$ and*

$$E_b=\sum_{(1,b)\subseteq(P,e)}[P,e].$$

Then the element E_b is a primitive idempotent of $\mathcal{O}X\Omega(G,\mathcal{P},G^c)$, and conversely any primitive idempotent of $\mathcal{O}X\Omega(G,\mathcal{P},G^c)$ has this form.

2 Symmetric group

(2.1) Brauer pairs for symmetric groups. We denote by $\mathbb{N}$ the positive integers and consider the product order in $\mathbb{N}\times\mathbb{N}$. If $(a,b)\in\mathbb{N}\times\mathbb{N}$ and $T\subseteq\mathbb{N}\times\mathbb{N}$ (Young tableau) we set

$$D_{(a,b)}(T)=T-\{(c,d)\in T\mid(a,b)\leq(c,d),\ (c+1,d+1)\notin T\}.$$

Let us denote by $S(E)$ the group of bijections of finite set E. We denote by e_T the p-block of $\mathcal{O}S(E)$ associated to irreducible character χ_T.

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(2.2) Theorem [Puig] (1987) [Pu87] *Let* $(a,b) \in \mathbb{N} \times \mathbb{N}$ *and set* $T' = D_{(a,b)}(T)$. *Assume that* $|T| \equiv |T'|$ *mod* p *and let* P' *be a* p*-subgroup of* $S(E)$ *such that* $P \subsetneq P'$ *and* $|E^{P'}| = |T'|$. *Then* $(P, Br_P(e_T)) \subsetneq (P', Br_{P'}(e_{T'}))$.

We can obtain a formula of p-blocks of $\mathcal{O}X\Omega(G,\mathcal{P},G^c)$ by using Puig's Theorem.

(2.3) EXAMPLE. $G = S_5$ and $p = 2$. Then the representatives of $C(\mathcal{P})$ are 1, $C_{2a} =< (4,5) >$, $C_{2b} =< (2,3)(4,5) >$, $E_{4a} =< (2,3)(4,5), (2,4)(3,5) >$, $C_4 =< (2,4,3,5) >$, $E_{4b} =< (2,3), (4,5) >$, $D_8 =< (4,5), (2,3), (2,4)(3,5) >$. Let $s = (1,2,3)$ and $t = (1,2,3,4,5)$. The idempotents of $\mathbb{K}X\Omega(G,\mathcal{P},G^c)$ are given as follows:

$$[1,b_1] = 11/1800(G/1)_1 - 1/90(G/1)_s + 1/75(G/1)_t,$$

$$[1,b_2] = 1/450(G/1)_1 + 1/90(G/1)_s - 1/75(G/1)_t,$$

$$[C_{2a},d_1] = -1/36(G/1)_1 - 1/18(G/1)_s + 1/18(G/C_{2a})_1 + 1/9(G/C_{2a})_s,$$

$$[C_{2a},d_2] = -1/18(G/1)_1 + 1/18(G/1)_s + 1/9(G/C_{2a})_1 - 1/9(G/C_{2a})_s,$$

$$[C_{2b},e_1] = -1/8(G/1)_1 + 1/4(G/C_{2b})_1,$$

$$[E_{4a},f_1] = 1/12(G/1)_1 - 1/4(G/C_{2b})_1 + 1/6(G/E_{4a})_1,$$

$$[C_4,g_1] = -1/4(G/C_{2b})_1 + 1/2(G/C_4)_1,$$

$$[E_{4b},h_1] = 1/4(G/1)_1 - 1/2(G/C_{2a})_1 - 1/4(G/C_{2b})_1 + 1/2(G/E_{4b})_1,$$

$$[D_8,i_1] = 1/2G/C_{2b})_1 - 1/2(G/E_{4a})_1 - 1/2(G/C_4)_1 - 1/2(G/E_{4b})_1 + (G/D_8)_1.$$

Thus 2-block idempotents of $\mathcal{O}X\Omega(G,\mathcal{P},G^c)$ given by Theorem (1.2) are

$$E_{b_1} = [1,b_1] + [C_{2a},d_1] + [C_{2b},e_1] + [E_{4a},f_1] + [C_4,g_1] + [E_{4b},h_1] + [D_8,i_1],$$

$$E_{b_2} = [1,b_2] + [C_{2a},d_2].$$

Moreover, the coefficients of the idempotents are now presented.

$$E_{b_1} = 14/75(G/1)_1 - 1/15(G/1)_s + 1/75(G/1)_t - 4/9(G/C_{2a})_1 + 1/9(G/C_{2a})_s - 1/3(G/E_{4a})_1 + (G/D_8)_1,$$

$$E_{b_2} = -14/75(G/1)_1 + 1/15(G/1)_s - 1/75(G/1)_t + 1/9(G/C_{2a})_1 - 1/9(G/C_{2a})_s.$$

The identity of $\mathcal{O} \otimes X\Omega(G,\mathcal{P},G^c)$ is

$$E_{b_1} + E_{b_a} = 2/15(G/1)_1 - 1/3(G/C_{2a}) - 1/3(G/E_{4a}) + (G/D_8).$$

References

[OY 01] F. Oda and T. Yoshida, Crossed Burnside rings I. The fundamental Theorem, *J. Algebra*, **236**, 29–79 (2001).

[OY 03] F. Oda and T. Yoshida, Crossed Burnside rings III, preparation (2003).

[Pu87] L. Puig, The Nakayama conjectures and the Brauer pairs, *Publ. Math. Univ. Paris VII*, **25**, 195–205 (1987).

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