Nonoscillation theorems for a class of fourth order quasilinear differential equations with deviating arguments

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Abstract

We consider the fourth-order quasilinear differential equation with deviating arguments

$$(*) \qquad (|y''(t)|^{\alpha}\operatorname{sgn} y''(t))'' + q(t)|y(t+\sin t)|^{\beta}\operatorname{sgn} y(t+\sin t) = 0,$$

where α, β are positive constants and $q : [0,\infty) \to (0,\infty)$ is a continuous function. In this paper we estabulish criteria for the existence of nonoscillatory solutions of $(*)$ which are classified into six disjoint classes according to their asymptotic behavior as $t \to \infty$.

0. Introduction

This paper is devoted to the study of the nonoscillatory behavior of fourth order quasilinear functional differential equations of the type

$$\text{(A)} \qquad (|y''(t)|^{\alpha}\operatorname{sgn} y''(t))'' + q(t)|y(t+\sin t)|^{\beta}\operatorname{sgn} y(t+\sin t) = 0,$$

where

(a) α and β are positive constants;

(b) $q : [0,\infty) \to (0,\infty)$ is a continuous function.

By a (positive) solution of (A) we mean a function $y \in C^2[T_y,\infty)$, $T_y \geq 0$, which has the property $|y''|^{\alpha}\operatorname{sgn} y'' \in C^2[T_y,\infty)$ and satisfies the equation for all sufficiently large $t \geq T_y$. Those solutions which vanish in a neighborhood of infinity will be excluded from our consideration. A solution is said to be oscillatory if it has a sequence of zeros clustering around ∞, and nonoscillatory otherewise.

We can show that six class of positive solutions of (A) are classified according to asymptotic behavior as $t \to \infty$. (see Wu [1].)

I_1: $\quad \displaystyle\lim_{t\to\infty} \frac{y(t)}{t^{2+\frac{1}{\alpha}}} = \text{const} > 0;$

I_2: $\quad \displaystyle\lim_{t\to\infty} \frac{y(t)}{t^{2+\frac{1}{\alpha}}} = 0, \quad \lim_{t\to\infty} \frac{y(t)}{t^2} = \infty;$

I_3: $\lim_{t\to\infty} \frac{y(t)}{t^2} = \text{const} > 0;$

II_1: $\lim_{t\to\infty} \frac{y(t)}{t} = \text{const} > 0;$

II_2: $\lim_{t\to\infty} \frac{y(t)}{t} = 0, \quad \lim_{t\to\infty} y(t) = \infty;$

II_3: $\lim_{t\to\infty} y(t) = \text{const} > 0.$

1. Existence of nonoscillatory solutions (necessary and sufficient conditions)

It will be shown that necessary and sufficient conditions can be established for the existence of positive solutions of the types I_1, I_3, II_1 and II_3.

Theorem 1.1. *There exists a positive solution of type* I_1 *of* (A) *if and only if*

$$\int_0^\infty t^{\left(2+\frac{1}{\alpha}\right)\beta} q(t)dt < \infty. \tag{1.1}$$

Proof. Let $y(t)$ be a solution of (A). Integrating (A) over $[T,\infty)$ gives

$$((y''(\infty))^{\alpha*})' - ((y''(T))^{\alpha*})' + \int_T^\infty q(s)(y(s+\sin s))^\beta ds = 0,$$

which impiles that

$$\int_T^\infty q(t)(y(t))^\beta dt < \infty.$$

This, combined with the asymptotic relation $\lim_{t\to\infty} y(t)/t^{2+\frac{1}{\alpha}} = \text{const} > 0$, shows that the condition (1.1) is satisfied.

Suppose now that (1.1) holds. Let $k > 0$ be any given constant. Choose $T > 0$ large enough so that

$$\left(\frac{\alpha^2}{(\alpha+1)(2\alpha+1)}\right)^\beta 2^{\left(2+\frac{1}{\alpha}\right)\beta} \int_T^\infty t^{\left(2+\frac{1}{\alpha}\right)\beta} q(t)dt \le \frac{(2k)^\alpha - k^\alpha}{(2k)^\beta}. \tag{1.2}$$

Put $T_* = \min\{T, \inf_{t\ge T}(t+\sin t)\}$, and define

$$G(t,T) = \int_T^t (t-s)(s-T)^{\frac{1}{\alpha}} ds = \frac{\alpha^2}{(\alpha+1)(2\alpha+1)}(t-T)^{2+\frac{1}{\alpha}}, \quad t \ge T,$$

(1.3)

$$G(t,T) = 0, \quad t \le T.$$

Let $Y \subset C[T_*,\infty)$ and $\mathcal{F} : Y \to C[T_*,\infty)$ be defined as follows:

$$Y = \{y \in C[T_*,\infty) : \ kG(t,T) \le y(t) \le 2kG(t,T), \ \ t \ge T_*\}, \tag{1.4}$$

$$\mathcal{F}y(t) = \int_T^t (t-s)\left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(y(\sigma+\sin\sigma))^\beta d\sigma\right) dr\right]^{\frac{1}{\alpha}} ds, \quad t \ge T$$

(1.5)

$$\mathcal{F}y(t) = 0, \quad T_* \le t \le T.$$

If $y \in Y$, then for $t \geq T$

$$\mathcal{F}y(t) \geq k\int_T^t (t-s)(s-T)^{\frac{1}{\alpha}}ds = kG(t,T)$$

and

$$\mathcal{F}y(t) \leq \int_T^t (t-s)\left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(2kG(\sigma+\sin\sigma,T))^\beta d\sigma\right)dr\right]^{\frac{1}{\alpha}}ds$$

$$\leq \int_T^t (t-s)\left[\int_T^s \left(k^\alpha + \left(\frac{\alpha^2\cdot 2k}{(\alpha+1)(2\alpha+1)}\right)^\beta 2^{(2+\frac{1}{\alpha})\beta}\int_r^\infty q(\sigma)\sigma^{(2+\frac{1}{\alpha})\beta}d\sigma\right)dr\right]^{\frac{1}{\alpha}}ds$$

$$\leq 2k\int_T^t (t-s)(s-T)^{\frac{1}{\alpha}}ds = 2kG(t,T),$$

and hence $\mathcal{F}y \in Y$. Thus, $\mathcal{F}$ maps Y into itself. Let $\{y_n\}$ be a sequence of functions in Y converging to $y \in Y$ in the metric topology of $C[T_*,\infty)$. Then, by using Lebesgue's dominated convergence theorem, we can prove that the sequence $\{\mathcal{F}y_n(t)\}$ converges to $\mathcal{F}y(t)$ as $n \to \infty$ uniformly on every compact interval of $[T_*,\infty)$, implying that $\mathcal{F}y_n \to \mathcal{F}y$ as $n \to \infty$ in $C[T_*,\infty)$. Hence $\mathcal{F}$ is a continuous mapping.
For any $y \in Y$ we have

$$(\mathcal{F}y(t))' = \int_T^t \left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(y(\sigma+\sin\sigma))^\beta d\sigma\right)dr\right]^{\frac{1}{\alpha}}ds, \quad t \geq T,$$

which implies that

$$0 \leq (\mathcal{F}(t))' \leq 2k\int_T^t (s-T)^{\frac{1}{\alpha}}ds = \frac{2k\alpha}{\alpha+1}(t-T)^{1+\frac{1}{\alpha}}, \quad t \geq T.$$

From this inequality, together with the fact that $\mathcal{F}y \in Y$, we conclude that the set $\mathcal{F}(Y)$ is relatively compact in the topology of $C[T_*,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem, there exists a fixed element $y \in Y$ of $\mathcal{F}$, i.e., $y = \mathcal{F}y$, which satisfies the integral equation

$$y(t) = \int_T^t (t-s)\left[\int_T^s \left(k^\alpha + \int_r^\infty q(\sigma)(y(\sigma+\sin\sigma))^\beta d\sigma\right)dr\right]^{\frac{1}{\alpha}}ds, \quad t \geq T. \tag{1.6}$$

Differentiation of (1.6) shows that $y(t)$ is a positive solution of (A) on $[T,\infty)$. Since $\lim_{t\to\infty}((y''(t))^\alpha)' = k^\alpha > 0$, $y(t)$ is a desired solution of type I_1. This completes the proof.

Theorem 1.2. *There exists a positive solution of type* I_3 *of* (A) *if and only if*

$$\int_0^\infty t^{2\beta+1}q(t)dt < \infty. \tag{1.7}$$

Theorem 1.3. *There exists a positive solution of type* II_1 *of* (A) *if and only if*

$$\int_0^\infty \left[\int_t^\infty (s-t)s^\beta q(s)ds\right]^{\frac{1}{\alpha}}dt < \infty. \tag{1.8}$$

Theorem 1.4. *There exists a positive solution of type* II_3 *of* (A) *if and only if*

$$\int_0^\infty t\left[\int_t^\infty (s-t)q(s)ds\right]^{\frac{1}{\alpha}}dt < \infty. \tag{1.9}$$

Theorems 1.2, 1.3 and 1.4 can be proved in a similar fashion.

2. Existence of nonoscillatory solutions (sufficient conditions)

This section is devoted to the study of positive solutions of types I_2 and II_2 of (A). Unlike the solutions of the types I_1, I_3, II_1 and II_3 it is impossible to characterize the existence of solutions of the types I_2 and II_2. We will therefore be content to obtain sufficient conditions for the existence of the such solutions. We observe that this kind of problem has not been dealt with even for ordinary differential equations without deviating arguments of the form

$$(|y''(t)|^{\alpha}\operatorname{sgn} y'')'' + q(t)|y(t)|^{\beta}\operatorname{sgn} y(t) = 0$$

(see Wu [1].)

Theorem 2.1. *There exists a positive solution of type* I_2 *of* (A) *if*

$$\int_0^{\infty} t^{\left(2+\frac{1}{\alpha}\right)\beta} q(t)dt < \infty \tag{2.1}$$

and

$$\int_0^{\infty} t^{2\beta+1} q(t)dt = \infty. \tag{2.2}$$

Theorem 2.2. *There exists a positive solution of type* II_2 *of* (A) *if*

$$\int_0^{\infty} \left[\int_t^{\infty} (s-t)s^{\beta} q(s)ds\right]^{\frac{1}{\alpha}} dt < \infty \tag{2.3}$$

and

$$\int_0^{\infty} t \left[\int_t^{\infty} (s-t)q(s)ds\right]^{\frac{1}{\alpha}} dt = \infty. \tag{2.4}$$

The proof of Theorems 2.1 and 2.2 will be omitted.

References

[1] F. Wu, Nonoscillatory solutions of fourth order quasilinear differential equations, Funkcialaj Ekvacioj, **45** (2002), 71 – 88.

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