A Note on Cohen-Macaulay Rings

KAGEYAMA Masaru*

Abstract

In the present paper, we introduce important classes of Noetherian local rings (regular local, Cohen-Macaulay, complete intersection...etc.) with some examples and we explain a relation among them as careful as possible.

Acknowledgement

The author, sincerely, would like to thank all the staffs working at Toyama national college of maritime technology for their kindness.

Let us start with an example of a regular local ring.

1 p-basis and a regular local ring

In this section, $(R, \mathfrak{m}, k)$ is a local domain with maximal ideal $\mathfrak{m}$ and the residue field k. We denote by R' an intermediate local ring such that $R^p \subset R' \subset R$, $\mathfrak{m}'$ the maximal ideal and k' the residue field of R'. Note that $k^p \subset k' \subset k$. When we say $\overline{A}$ is p-*independent*, we tacitly assume that A maps injectively to $\overline{A}$.

Lemma 1.1 (Lemma 2.4 [1]) *Let A be a subset of R. If A is p-independent over R' and $\overline{A}$ is p-independent over k', then a minimal system of generators for $\mathfrak{m}'$ is also a minimal system of generators for the maximal ideal of $R'[A]$. In addition, R' is regular, so is $R'[A]$.*

Proof. Let us prove that if R' is a regular local ring then so is $R'[A]$. It is clear that $R'[A]$ is integral over R'. Hence $\dim R'[A] = \dim R'$ and it follows from the earlier result that:

$$\begin{aligned}\operatorname{rank} R'[A] \cap \mathfrak{m}/(R'[A] \cap \mathfrak{m})^2 &= \operatorname{rank} \mathfrak{m}'/\mathfrak{m}'^2 \\ &= \dim R' \\ &= \dim R'[A]. \square\end{aligned}$$

Lemma 1.2 (Lemma 2.5 [1]) *Let A be a subset of R. If R' is regular and $\overline{A}$ is p-independent over k', then A is p-independent over K'.*

By using the above lemmas, we have the following:

Theorem 1.3 (Theorem 2 [2]) *Let R be a regular local ring of characteristic $p > 0$, k be the residue field, A be a subset of R. If A is a system of representative of a p-basis of k over k^p, then $R^p[A]$ is regular local.*

Proof. By assumption, $\overline{A}$ is p-independent over k^p. By Lemma 1.2, A is p-independent over R^p. Applying Lemma 1.1, we get our assertion.□

Here we will enumerate the following basic results:

Example: 1.4 Let R be a Noetherian local ring.

*一般教養科（講師）数学
E-mail: kageyama@nc-toyama.ac.jp

 （平成22年３月31日受付）

(1) $\dim R = 0$ and R is regular
$\iff R$ is a field.

(2) $\dim R = 1$ and R is regular
$\iff R$ is a DVR.

(3) n-variable formal power series $k[\![X_1, \ldots, X_n]\!]$ over a field k, is regular local and so is $k[X_1, \ldots, X_n]_{\mathfrak{m}}$ where $\mathfrak{m} = (X_1, \ldots, X_n)$.

2 Cohen-Macaulay, Gorenstein and complete intersection rings

In this section, we will introduce Noetherian local rings which are well known. Let us begin with the definition of a CM local ring.

Definition 2.1 Let A be a Noetherian local ring. We say that A is *Cohen-Macaulay* (briefly, CM) if $\mathrm{depth} A = \dim A$.

The following two results explain us the term Cohen-Macaulay.

Example: 2.2 (1) The ring of polynomials $k[X]$ over a field k is a CM ring. This was proved by Macaulay in 1916.

(2) Every regular local ring is a CM ring. This result was proved by I.S.Cohen in 1946.

Next we will introduce basic results about CM rings.

Example: 2.3 (1) A 0-dimensional Noetherian ring is a CM ring.

(2) A reduced 1-dimensional Noetherian ring is a CM ring.

(3) A 2-dimensional normal ring is a CM ring.

In the following proofs, we denote by A a Noetherian ring.

Proof of (1): By [3, Theorem 17.6], it is enough to show that (0) is unmixed. So, we must prove that (0) has no embedded prime divisors. But it is clear because $\dim A = 0$.□

Proof of (2): We only need to prove that (0) and every principle ideal (a_1) of A such that $\mathrm{ht}(a_1) = 1$ are unmixed.

Since $\sqrt{(0)} = (0)$, (0) has no embedded prime divisors, so (0) is unmixed.

If (a_1) has an embedded prime divisor $\mathfrak{p}$, then there is a minimal prime divisor $\mathfrak{p}'$ of (a_1) such that $(a_1) \subset \mathfrak{p}' \subsetneq \mathfrak{p}$. Hence $\mathrm{ht}(\mathfrak{p}') = 0$ but this is a contradiction.□

Proof of (3): We will prove $A_{\mathfrak{m}}$ is a CM local ring for all maximal ideals $\mathfrak{m}$ of A.

If $\mathrm{ht}(\mathfrak{m}) = 0$ then $\dim A_{\mathfrak{m}} = \mathrm{ht}(\mathfrak{m}) = 0$. It follows from (1).

If $\mathrm{ht}(\mathfrak{m}) = 1$ then $\dim A_{\mathfrak{m}} = \mathrm{ht}(\mathfrak{m}) = 1$ and $A_{\mathfrak{m}}$ is domain, that is $A_{\mathfrak{m}}$ is reduced. It follows from (2).

Finally, we suppose $\mathrm{ht}(\mathfrak{m}) = 2$. Then $A_{\mathfrak{m}}$ is reduced and its zero ideal has no embedded prime divisors. So it is unmixed. Since $(A_{\mathfrak{m}})_{\mathfrak{p}A_{\mathfrak{m}}} = A_{\mathfrak{p}}$, A is an integrally closed domain. □

Remark: 2.4 Let $k[X, Y]$ be the ring of two-variable polynomials over a field k and set $A = k[X, Y]/(X^2, XY)$. Then A is 1-dimensional Noetherian ring but it is not a CM ring.

It is a well known result that the following conditions are equivalent.

Theorem 2.5 (Theorem 18.1 [3]) *Let $(A, \mathfrak{m}, k)$ be an n-dimensional Noetherian local ring. Then the following conditions are equivalent:*

(1) inj $\dim A < \infty$*;*

(1') inj $\dim A = n$*;*

(2) $\mathrm{Ext}^i_A(k, A) = 0$ *for* $i \neq n$ *and* $\simeq k$ *for* $i = n$*;*

(3) $\mathrm{Ext}^i_A(k, A) = 0$ *for some* $i > n$*;*

(4) $\mathrm{Ext}^i_A(k, A) = 0$ *for all* $i < n$ *and* $\simeq k$ *for* $i = n$*;*

(4') A is a CM ring and $\mathrm{Ext}^n_A(k, A) \simeq k$*;*

(5) A is a CM ring and every parameter ideal of A is irreducible;

(5') A is a CM ring and there exists an irreducible parameter ideal.

Now we come to the definition of a Gorenstein local ring.

Definition 2.6 A Noetherian local ring for which the above equivalent conditions hold is said to be *Gorenstein.* A Noetherian ring A is *Gorenstein* if its localisation at every maximal ideal is a Gorenstein local ring.

There is an interesting class of Noetherian ring as follows:

Remark: 2.7 (Definition 2.5 [4])
A Noetherian ring A is said to be *acceptable* if the following three conditions are satisfied.

(a) A is universally catenary;

(b) whenever B is a finitely generated A-algebra, the Gorenstein locus of A is open in $\mathrm{Spec}(B)$; and

(c) for every prime ideal $\mathfrak{p}$ of A, the natural flat ring homomorphism $A_{\mathfrak{p}} \to (A_{\mathfrak{p}})\widehat{}$ is a Gorenstein ring homomorphism.

The notion of the acceptable ring is defined by R.Y.Sharp. It is analogous of an excellent ring.

Remark: 2.8 It is clear from the definition that every Gorenstein ring is CM.

Next we will define a complete intersection ring.

Lemma 2.9 *Let A be a ring and*

$$C_\bullet : \cdots \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \xrightarrow{d_{i-1}} \cdots$$

a complex of A-modules. For any A-module N we denote by $C_\bullet \otimes N$ *for the complex*

$$\cdots \xrightarrow{d_{i+1}\otimes 1} C_i\otimes N \xrightarrow{d_i\otimes 1} C_{i-1}\otimes N \xrightarrow{d_{i-1}\otimes 1} \cdots .$$

If N is flat over A then

$$H_i(C_\bullet) \otimes N = H_i(C_\bullet \otimes N).$$

Proof. We note that $H_i(C_\bullet) = \ker d_i/\mathrm{im}\ d_{i+1}$. Since N is A-flat, we have an exact sequence $0 \to \mathrm{im}\ d_{i+1} \otimes N \to \ker d_i \otimes N \to (\ker d_i/\mathrm{im}\ d_{i+1}) \otimes N \to 0$.

Hence it follows that

$$\begin{aligned} H_i(C_\bullet \otimes N) &= \ker(d_i \otimes 1)/\mathrm{im}\ (d_{i+1} \otimes 1) \\ &= \ker d_i \otimes N/\mathrm{im}\ d_{i+1} \otimes N \\ &= (\ker d_i/\mathrm{im}\ d_{i+1}) \otimes N \\ &= H_i(C_\bullet) \otimes N. \square \end{aligned}$$

Let us consider the case when a regular local ring $(A, \mathfrak{m}, k)$ can be expressed as a quotient of a regular local ring $(R, \mathfrak{n})$. Let M be an R-module. We denote by $\mu(M)$ for the minimum number of generators of M. Then it follows that

$$\begin{aligned} \mu(\mathfrak{a}) &= \dim_k \mathrm{Tor}_1^R(k, A) \\ &= \dim_k H_1(E_\bullet) \\ &= \epsilon_a(A). \end{aligned}$$

Theorem 2.10 *Let* $(A, \mathfrak{m}, k)$ *be a Noetherian local ring and* $\widehat{A}$ *be its completion.*

(i) $\epsilon_p(A) = \epsilon_p(\widehat{A})$ *for all* $p \geq 0$*,*

(ii) $\epsilon_1(A) \geq \mathrm{em}\ \dim A - \dim A$*,*

(iii) If R is a regular local ring, $\mathfrak{a}$ *is an ideal of R and* $A \simeq R/\mathfrak{a}$ *then*

$$\mu(\mathfrak{a}) = \dim R - \mathrm{em}\ \dim A + \epsilon_1(A).$$

Proof. (i) Since $\mathfrak{m}/\mathfrak{m}^2 \simeq \widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2$ and $A/\mathfrak{m} \simeq \widehat{A}/\widehat{m} = k$, so that a minimal basis of $\mathfrak{m}$ is a minimal basis of $\widehat{\mathfrak{m}}$. Then it follows that $\mathrm{rank}_k(\mathfrak{m}/\mathfrak{m}^2) = \mathrm{rank}_k(\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2)$. Since

$$\begin{aligned} \bigwedge(Ae_1 \oplus \cdots \oplus Ae_n) \otimes_A \widehat{A} &= \bigwedge(A^n \otimes_A \widehat{A}) \\ &= \bigwedge \widehat{A}^n, \end{aligned}$$

by tensoring $C_\bullet$ through with $\widehat{A}$, we have a Koszul complex induced from a minimal basis of $\widehat{\mathfrak{m}}$. Since $\widehat{A}$ is A-flat, by using Lemma 2.9, we have

$$H_p(E_\bullet) \otimes_A \widehat{A} = H_p(E_\bullet \otimes_A \widehat{A}).$$

On the other hand

$$\begin{aligned} H_p(E_\bullet) \otimes_A \widehat{A} &= \widehat{H_p(E_\bullet)} \\ &= \varprojlim H_p(E_\bullet)/\mathfrak{m}^n H_p(E_\bullet) \\ &= H_p(E_\bullet). \end{aligned}$$

Thus $H_p(E_\bullet) \simeq H_p(E_\bullet \otimes_A \widehat{A})$. We get the assertion.

(ii) Let us consider the case when A is expressed as a quotient of a regular local ring $(R, \mathfrak{n})$. Then we may assume that there exists an ideal $\mathfrak{a}$ of R such that $A \simeq R/\mathfrak{a}$ and $\mathfrak{a} \subset \mathfrak{n}^2$. Then

$$\begin{aligned} \epsilon_1(A) &= \mu(\mathfrak{a}) \\ &\geq \mathrm{ht}(\mathfrak{a}) \ \ \text{(by [3, Theorem 13.5])} \\ &= \dim R - \dim(R/\mathfrak{a}) \ \ \text{(by } R \text{ is CM)} \\ &= \mathrm{em}\ \dim A - \dim A. \end{aligned}$$

We have to prove in general case, however it is follows from [3, Theorem 29.4] that $\widehat{A}$ always is a quotient of a regular local ring. By (i), $\epsilon_1(A) = \epsilon_1(\widehat{A})$, em $\dim \widehat{A}$ = em $\dim A$ and $\dim \widehat{A} = \dim A$, hence, we get the assertion.

(iii) Let $\mathfrak{n}$ be the maximal ideal of R. If $\mathfrak{a} \subset \mathfrak{n}^2$, then $\mu(\mathfrak{a}) = \epsilon_1(A)$ and $\dim R$ = em $\dim A$, we obtain our assertion. If $\mathfrak{a} \not\subset \mathfrak{n}^2$, then we can chose an element $x \in \mathfrak{a} \backslash \mathfrak{n}^2$. We will show the following calm:

Claim 2.11

$$\dim R/xR = \dim R - 1, \mu(\mathfrak{a}/xR) = \mu(\mathfrak{a}) - 1.$$

Proof. The first assertion is clear. We will prove the second assertion. Since $\mathfrak{a}$ is generated by $\mu(\mathfrak{a}/xR) + 1$ elements. Therefore $\mu(\mathfrak{a}/xR) + 1 \geq \mu(\mathfrak{a})$. On the other hand, $\overline{x} \neq 0$ in $\mathfrak{a}/\mathfrak{n}\mathfrak{a}$. Then we can take a k-basis $\overline{x}, \overline{a}_1, \ldots, \overline{a}_{n-1}$ ($a_i \in \mathfrak{a}, n = \mu(\mathfrak{a})$) and the images of $a_1, \ldots, a_{n-1}$ generate $\mathfrak{a}/xR$ as an R-module. Thus $\mu(\mathfrak{a}/xR) \leq n - 1 = \mu(\mathfrak{a}) - 1$.

Let us prove the assertion by induction on $\dim R = r$. If $r = 0$ then R is a field, so we are done. When $r > 0$, since $\dim(R/xR) = r - 1$, by inductive hypothesis

$$\mu(\mathfrak{a}/xR) = \dim(R/xR) - \mathrm{em}\ \dim A + \epsilon_1(A).$$

By using the above claim, $\mu(\mathfrak{a}) = \dim R -$ em $\dim A + \epsilon_1(A)$.□

We are now ready to define a complete intersection ring.

Definition 2.12 Let A be a Noetherian local ring. We say that A is *complete intersection* (briefly, C.I.) if $\epsilon_1(A)$ = em $\dim A - \dim A$.

For example, it is known the following:

Example: 2.13 (1) Every regular local ring is C.I.

(2) Let A be a Noetherian local ring. If A is CM and em $\dim A = \dim A + 1$ then A is C.I.

(3) Let k be a field and set $A = k[\![X, Y, Z]\!]/(X^2 - Y^2, Y^2 - Z^2, XY, YZ, ZX)$ then A is Gorenstein but is not C.I.

Here let us show (2).

Proof of (2): We may assume that A is a complete local ring. Then by using [3, Theorem 29.4], there exist a regular local ring $(R, \mathfrak{n})$ and an ideal I of R such that $I \subset \mathfrak{n}^2$ and $A \simeq R/I$. Then $\dim R$ = em $\dim A$ and $\dim A = \dim(R/I) = \dim R - \mathrm{ht}(I)$.

By the assumption, we have $\mathrm{ht}(I) = 1$. This means that $\mathrm{ht}(0) = 1$ in A. Since A

is a CM ring, (0) is unmixed. Thus $\mathrm{ht}(\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \mathrm{Ass}(A/I)$.

Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be a shortest primary decomposition. Since $\mathrm{ht}(\mathfrak{q}_i) = 1$ and R is a UFD, every $\mathfrak{q}_i$ is a principle ideal. Hence so is I. By the assumption $I \neq (0)$. Thus $\epsilon_1(A) = \mu(I) = \mathrm{em}\ \mathrm{dim} A - \dim A$ and this means A is a C.I. ring.□

3 A chain of implications for Noetherian local rings

In the last section we will discuss a relation among Noetherian local rings which are defined previous sections.

At first, we can show that every C.I. ring is Gorenstein.

Theorem 3.1 *Let A be a Noetherian local ring.*

(i) A is C.I. $\Longleftrightarrow$ $\widehat{A}$ is C.I.

(ii) Let A be a C.I. ring and R a regular local ring such that $A = R/\mathfrak{a}$; then $\mathfrak{a}$ is generated by an R-sequence. Conversely, if $\mathfrak{a}$ is an ideal generated by an R-sequence then $R/\mathfrak{a}$ is a C.I. ring.

(iii) A is C.I. if and only if the completion $\widehat{A}$ should be a quotient of a complete regular local ring R by an ideal generated by an R-sequence.

Proof. (i) By the above theorem, $\epsilon_1(A) = \epsilon_1(\widehat{A})$ since $\mathrm{em}\ \mathrm{dim} A = \mathrm{em}\ \mathrm{dim} \widehat{A}$ and $\dim A = \dim \widehat{A}$, hence (i) follows.

(ii) If A is C.I. and R is a regular local ring such that $A = R/\mathfrak{a}$ for an ideal $\mathfrak{a}$ of R. By Theorem 2.10 $\mu(\mathfrak{a}) = \dim R - \mathrm{em}\ \mathrm{dim} A + \epsilon_1(A)$. Since A is C.I. $\epsilon_1(A) = \mathrm{em}\ \mathrm{dim} A - \dim A$,

$$\begin{aligned}\mu(\mathfrak{a}) &= \dim R - \dim A \\ &= \dim R - \dim R/\mathfrak{a} \\ &= \dim R - (\dim R - \mathrm{ht}(\mathfrak{a})) \\ &\qquad \text{(because } R \text{ is regular)} \\ &= \mathrm{ht}(\mathfrak{a}).\end{aligned}$$

Thus, by [3, Theorem 17.4], a minimal generators of $\mathfrak{a}$ is an R-sequence.

Conversely, we suppose that $\mathfrak{a}$ is generated by an R-sequence $a_1, \ldots, a_r \in R$. By [3, Theorem 17.4], $\mathrm{ht}(\mathfrak{a}) = r$ and by Krull's altitude theorem, $r \leq \mu(\mathfrak{a})$. It is clear that $r \geq \mu(\mathfrak{a})$. Hence we have $\mathrm{ht}(\mathfrak{a}) = \mu(\mathfrak{a})$. Using Theorem 3.1 (iii)

$$\mu(\mathfrak{a}) = \dim R - \mathrm{em}\ \mathrm{dim}(R/\mathfrak{a}) + \epsilon_1(R/\mathfrak{a}).$$

Hence

$$\begin{aligned}\epsilon_1(R/\mathfrak{a}) &= \mathrm{em}\ \mathrm{dim}(R/\mathfrak{a}) - (\dim R - \mathrm{ht}(\mathfrak{a})) \\ &= \mathrm{em}\ \mathrm{dim}(R/\mathfrak{a}) - (\dim R/\mathfrak{a}).\end{aligned}$$

Thus $R/\mathfrak{a}$ is also C.I.

(iii) $\Longleftarrow$) Let $\mathfrak{a}$ be an ideal of R generated by an R-sequence such that $R/\mathfrak{a} \simeq \widehat{A}$. By (ii), $\widehat{A}$ is C.I. Hence, by (i), so is A.

$\Longrightarrow$) If A is C.I. then so is $\widehat{A}$. By [3, Theorem 29.4], $\widehat{A}$ is a homomorphic image of a complete regular local ring R. Thus it follows from (ii) that the kernel is generated by an R-sequence.□

Theorem 3.2 *A C.I. ring is Gorenstein.*

Proof. Let A be a C.I. ring. By Theorem 3.1(i), if A is C.I. then so is $\widehat{A}$. If $\widehat{A}$ is Gorenstein then so is A. Thus we may assume that A is complete.

By Theorem 3.1(ii), there exist a regular local ring R and an ideal $\mathfrak{a}$ generated by an R-sequence such that $A = R/\mathfrak{a}$. Since the maximal ideal of R is generated by a system of parameters and irredeucible, so that R is

Gorenstein. Here we need the following two claims:

Claim 3.3 *Let $(A, \mathfrak{m})$ be a Noetherian local ring, $M \neq 0$ a finite A-module and $a_1, \ldots, a_r \in \mathfrak{m}$ an M-sequence. We set $M' = M/(a_1, \ldots, a_r)M$. Then $\dim M' = \dim M - r$.*

Proof. It is sufficient to show for $r = 1$. The case $\dim M' \leq \dim M - 1$ follows from dimension theory. Let us prove that $\dim M' \geq \dim M - 1$. We set $a = a_1$ and $I = \mathrm{ann}_A(M/aM)$ and $s = \dim A/I$. Then there exist $x_1, \ldots, x_s \in \mathfrak{m}$ such that $\sqrt{(\overline{x}_1, \ldots, \overline{x}_s)} = \mathfrak{m}/I$. On the other hand

$$\begin{aligned} \sqrt{(\overline{x}_1, \ldots, \overline{x}_s)} &= \sqrt{(x_1, \ldots, x_s) + I/I} \\ &= \sqrt{(x_1, \ldots, x_s) + I}/I. \end{aligned}$$

Thus, we have $\sqrt{(x_1, \ldots, x_s) + I} = \mathfrak{m}$.

Since $\sqrt{I} = \sqrt{\mathrm{ann}_A(M) + (a)}$, so

$$\begin{aligned} &\sqrt{(a, x_1, \ldots, x_s) + \mathrm{ann}_A(M)} \\ &\qquad = \sqrt{(x_1, \ldots, x_s) + I}. \end{aligned}$$

Therefore $\sqrt{(a, x_1, \ldots, x_s) + \mathrm{ann}_A(M)} = \mathfrak{m}$, so that $(\overline{a}, \overline{x}_1, \ldots, \overline{x_s})$ is a $(\mathfrak{m}/\mathrm{ann}_A(M))$-primary ideal. Hence $s + 1 \geq \dim M$.

Claim 3.4 *Let $(A, \mathfrak{m})$ be a Noetherian local ring, $x_1, \ldots, x_n$ an A-sequence and we put $B = A/(x_1, \ldots, x_n)$. Then*

A is Gorenstein $\Longleftrightarrow$ B is Gorenstein.

Proof. $\Longrightarrow$) If A is Gorenstein, $\mathrm{Ext}^i_A(k, A) = 0$ for some $i > n$. By [3, Lemma 2], $\mathrm{Ext}^{i-r}_B(k, B) = \mathrm{Ext}^i_A(k, A) = 0$ and $i - r > n - r$. Hence we get the assertion.

$\Longleftarrow$) If B is Gorenstein, $\mathrm{Ext}^j_B(k, B) = 0$ for some $j > n - r$. Since $\mathrm{Ext}^{j+r}_A(k, A) = \mathrm{Ext}^j_B(k, B) = 0$, A is Gorenstein. □

We have a relation as follows:

Regular ⇒ C.I. ⇒ Gorenstein ⇒ CM.

References

[1] T. Kimura and H. Niitsuma, *Regular local rings of characteristic p and p-basis*, J. Math. Soc. Japan, Vol. 32, No.2, 1980, pp.363-371.

[2] T. Kimura, *p-basis of a regular local ring of characteristic p*, TRU Mathematics, 19-1, 1983, pp.33-46.

[3] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[4] R. Y. Sharp, *Acceptable Rings and Homomorphic Images of Gorenstein Rings*, J. Algebra, 44, pp.246-261(1977).